MATH3705 A — Test 1

Name and Student Number:

Total points: 20. No partial marks for Questions 1-4.

Closed book! Formula Sheet and Non-programmer calculators are allowed!

[**2**] 1. $L\{t \sin 3t\} =$

(a) $\frac{3s}{(s^2+3)^2}$ (b) $\frac{3s}{s^2+9}$ (c) $\frac{2\sin s}{(s^2+9)^2}$ (d) $-\frac{2e^s}{(s^2+3)^2}$ (e) $\frac{6s}{(s^2+9)^2}$

Solution: (e)
$L\{\sin 3t\} = \frac{3}{s^2+9}, \Rightarrow L\{t \sin 3t\} = -\left(\frac{3}{s^2+9}\right)' = \frac{6s}{(s^2+9)^2}.$

[**2**] 2. Find $L\{f(t)\}$, where $f(t) = 2t^3 - e^{-3t}\cos(4t)$.

(a) $\frac{12}{s^4} - \frac{(s+3)}{(s+3)^2+4}$ (b) $\frac{6}{s^4} - \frac{(s-3)}{(s-3)^2+16}$ (c) $\frac{12}{s^4} - \frac{(s-3)}{(s-3)^2+16}$
(d) $\frac{6}{s^4} - \frac{(s+3)}{(s+3)^2+16}$ (e) $\frac{12}{s^4} - \frac{(s+3)}{(s+3)^2+16}$

Solution: (e)
By linearity of LT and the First Shift Theorem, we have

$$F(s) = \frac{2(3!)}{s^4} - \frac{(s+3)}{(s+3)^2+16}.$$

[**2**] 3. Let $f(t)$ be 2-periodic for $t \geq 0$, and $f(t) = \begin{cases} 0, & 0 \leq t < 1; \\ t, & 1 \leq t < 2. \end{cases}$ Find $L\{f(t)\}$.

(a) $\frac{(1+s)e^{-s}-(1+2s)e^{-2s}}{s^2(1-e^{-2s})}$ (b) $\frac{se^{-s}-(1+2s)e^{-2s}}{s^2(1-e^{-2s})}$ (c) $\frac{(1+s)e^{-s}-(2s)e^{-2s}}{s^2(1-e^{-2s})}$
(d) $\frac{se^{-s}-(2s)e^{-2s}}{s^2(1-e^{-2s})}$ (e) $\frac{(s-1)e^{-s}-(1+2s)e^{-2s}}{s^2(1-e^{-2s})}$

Solution: (a)

$$L\{f(t)\} = \frac{1}{1-e^{-2s}}\int_0^2 f(t)e^{-st}dt = \frac{1}{1-e^{-2s}}\int_1^2 te^{-st}dt.$$

Note that, by integration-by-parts,

$$\int te^{-st}dt = \int td\frac{e^{-st}}{-s} = \frac{te^{-st}}{-s} - \int \frac{e^{-st}}{-s}dt = -\frac{te^{-st}}{s} - \frac{e^{-st}}{s^2} + C,$$

we have

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-2s}}\left[-\frac{te^{-st}}{s} - \frac{e^{-st}}{s^2}\right]|_1^2 = \frac{(1+s)e^{-s} - (1+2s)e^{-2s}}{s^2(1-e^{-2s})}.$$

[2] 4. Calculate $L\{t\sqrt{t}\}$.

(a) $\frac{3\sqrt{\pi}}{4s^{5/2}}$ (b) $\frac{3\pi}{4s^{5/2}}$ (c) $\frac{3}{4\sqrt{\pi}s^{5/2}}$ (d) $\frac{3\sqrt{\pi}}{s^{5/2}}$ (e) $\frac{\pi}{4s^{5/2}}$

Solution: (a)

$$L\{t\sqrt{t}\} = L\{t^{\frac{3}{2}}\} = \frac{\Gamma(\frac{5}{2})}{s^{5/2}}$$

$$= \frac{\frac{3}{4}\sqrt{\pi}}{s^{5/2}} = \frac{3\sqrt{\pi}}{4s^{5/2}}.$$

[6] 5. Using the Laplace transform solve the differential equation

$$f'' + f' - 6f = 70e^{4t}$$

with boundary conditions $f(0) = f'(0) = 0$.

Solution: Applying Laplace transform to the two sides, we get the subsidiary equation

$$s^2F + sF - 6F = \frac{70}{s-4}$$

or

$$F(s) = \frac{70}{(s-4)(s+3)(s-2)}$$

By partial fraction

$$\begin{aligned}\frac{70}{(s-4)(s+3)(s-2)} &= \frac{A}{s-4} + \frac{B}{s+3} + \frac{C}{s-2} \\ 70 &= A(s+3)(s-2) + B(s-4)(s-2) + C(s+3)(s-4).\end{aligned}$$

To find A, B, C, we use special s values: $s = 4$ gives $A = 5$, and $s = -3$ gives $B = 2$, and $s = 2$ gives $C = -7$. Putting all this together we have

$$F(s) = \frac{5}{s-3} + \frac{2}{s+3} + \frac{-7}{s-2},$$

Thus

$$f(t) = 5e^{3t} + 2e^{-3t} - 7e^{2t}.$$

[6] 6. Find $L^{-1}\left\{\frac{-5e^{-3s}}{s(s^2+4s+5)}\right\}$.

Solution: Let $G(s) = \frac{-5}{s(s^2+4s+5)}$. By partial fraction, we have

$$G(s) = \frac{-1}{s} + \frac{s+4}{s^2+4s+5} = -\frac{1}{s} + \frac{s+2}{(s+2)^2+1} + \frac{2}{(s+2)^2+1}.$$

Thus

$$g(t) = -1 + e^{-2t}\cos(t) + 2e^{-2t}\sin(t).$$

By the Second Shift Theorem,

$$f(t) = u(t-3)\left[-1 + e^{-2(t-3)}\cos(t-3) + 2e^{-2(t-3)}\sin(t-3)\right].$$